Exercise 1.2

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1. *Show that the function* $f : \mathbf{R}_* \to \mathbf{R}_*$ *defined by* $f(x) = \frac{1}{x}$ *is one-one and onto, where* $\mathbf{R}_*$ *is the set of all non-zero real numbers. Is the result true, if the domain* $\mathbf{R}_*$ *is replaced by* $\mathbf{N}$ *with co-domain being same as* $\mathbf{R}_*$ *?*

Solution :

Domain: R_* *is the set of all non-zero real numbers.*

Codomain: R_* *is the set of all positive real numbers.*

Injectivity (One-one):

A function f is injective if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

- *Assume $f(x_1) = f(x_2)$.*
- *Then, $\frac{1}{x_1} = \frac{1}{x_2}$.*
- *By cross-multiplying, we get $x_2 = x_1$.*

Therefore, $f(x) = \frac{1}{x}$ is injective.

Surjectivity (Onto):

A function f is surjective if for every element y in the codomain, there is an x in the domain such that $f(x) = y$.

For any $y \in \mathbf{R}_*$, we need to find an $x \in \mathbf{R}_*$ such that $\frac{1}{x} = y$.

- *Solving for x, we get $x = \frac{1}{y}$, which is a non-zero real number.*

Thus, every positive real number y has a pre-image $x = \frac{1}{y}$ in $\mathbf{R}_*$. Therefore, $f(x) = \frac{1}{x}$ is surjective.

If the domain $\mathbf{R}_*$ is replaced by $\mathbf{N}$ (the set of natural numbers), and the codomain remains $\mathbf{R}_*$:

- *The function $f(x) = \frac{1}{x}$ where x is a natural number will always give a positive real number.*
- *However, not every positive real number can be written as $\frac{1}{x}$ for some natural number x because $\frac{1}{x}$ for natural x is always a rational number of the form $\frac{1}{n}$ where n is a natural number. Thus, it is not surjective onto the entire set of positive real numbers.*

2. Check the injectivity and surjectivity of the following functions:

(i) $f : \mathbf{N} \to \mathbf{N}$ given by $f(x) = x^2$

Solution :

- ***Injectivity***: *Assume $f(x_1) = f(x_2)$. Then $x_1^2 = x_2^2$. Since x_1 and x_2 are natural numbers, this implies $x_1 = x_2$.*
- ***Surjectivity***: *Not every natural number is a perfect square. For example, 2 or 3 is not a perfect square. Thus, it is not surjective.*

(ii) $f : \mathbf{Z} \to \mathbf{Z}$ given by $f(x) = x^2$

Solution :

- ***Injectivity***: *$x_1^2 = x_2^2$ implies $x_1 = \pm x_2$. Therefore, it is not injective.*
- ***Surjectivity***: *Not every integer is a perfect square. For example, -1 is not a perfect square. Thus, it is not surjective.*

(iii) $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2$

Solution :

- ***Injectivity***: *$x_1^2 = x_2^2$ implies $x_1 = \pm x_2$. Therefore, it is not injective.*
- ***Surjectivity***: *Not every real number is a perfect square. For example, negative numbers are not perfect squares. Thus, it is not surjective.*

(iv) $f : \mathbf{N} \to \mathbf{N}$ given by $f(x) = x^3$

Solution :

- ***Injectivity***: *Assume $f(x_1) = f(x_2)$. Then $x_1^3 = x_2^3$, which implies $x_1 = x_2$. Therefore, it is injective.*

- ***Surjectivity***: *Not every natural number is a perfect cube. For example, 2 or 4 is not a perfect cube. Thus, it is not surjective.*

(v) $f : \mathbf{Z} \to \mathbf{Z}$ *given by* $f(x) = x^3$

Solution :

- ***Injectivity***: $x_1^3 = x_2^3$ *implies* $x_1 = x_2$. *Therefore, it is injective.*
- ***Surjectivity***: *Every integer can be expressed as a cube of some integer. Therefore, it is surjective.*

3. Prove that the Greatest Integer Function $f : \mathbf{R} \to \mathbf{R}$, *given by* $f(x) = [x]$, *is neither one-one nor onto.*

Solution :

- ***Not one-one***: *For any interval* $[n, n+1)$ *where* n *is an integer, the function* $f(x) = n$ *for all* x *in this interval. So,* $f(1.5) = f(1.9) = 1$, *but* $1.5 \neq 1.9$. *Therefore,* f *is not injective.*
- ***Not onto***: *The function* $f(x) = [x]$ *only maps to integers, so it is not surjective onto* $\mathbf{R}$, *the set of all real numbers.*

4. Show that the Modulus Function $f : \mathbf{R} \to \mathbf{R}$, *given by* $f(x) = |x|$, *is neither one-one nor onto.*

Solution :

- ***Not one-one***: *For* $x = 2$ *and* $x = -2$, $f(2) = f(-2) = 2$, *but* $2 \neq -2$. *Therefore,* f *is not injective.*
- ***Not onto***: *The function* $f(x) = |x|$ *maps to non-negative real numbers only, so it cannot map to negative real numbers. Therefore,* f *is not surjective.*

5. *Show that the Signum Function* $f : \mathbf{R} \to \mathbf{R}$, *given by*

$$f(x) = \begin{cases} 1, & if\ x > 0 \\ 0, & if\ x = 0 \\ -1, & if\ x < 0 \end{cases}$$

is neither one-one nor onto.

Solution :

- ***Not one-one***: *For* $x = 1$ *and* $x = 2$, $f(1) = f(2) = 1$, *but* $1 \neq 2$. *Similarly, for* $x = -1$ *and* $x = -2$, $f(-1) = f(-2) = -1$, *but* $-1 \neq -2$. *Therefore,* f *is not injective.*
- ***Not onto***: *The function* f *maps to* $\{-1, 0, 1\}$, *but the codomain here is* $\mathbf{R}$, *so it is not surjective.*

6. *Let* $\mathrm{A} = \{1, 2, 3\}$ *and* $\mathrm{B} = \{4, 5, 6, 7\}$. *Let* $f = \{(1, 4), (2, 5), (3, 6)\}$ *be a function from A to B. Show that* f *is one-one.*

Solution :

- ***Injectivity***: *Each element of* A *is mapped to a distinct element in* B. *Thus,* f *is injective.*

7. *In each of the following cases, state whether the function is one-one, onto, or bijective. Justify your answer.*

   (i) $f : \mathbf{R} \to \mathbf{R}$ *defined by* $f(x) = 3 - 4x$

Solution :

- ***Injectivity (One-one)***:

  To check if the function is injective, assume $f(x_1) = f(x_2)$.

$$3 - 4x_1 = 3 - 4x_2 \implies -4x_1 = -4x_2 \implies x_1 = x_2$$

  Hence, $f(x)$ *is injective.*

- ***Surjectivity (Onto)***:

  To check if the function is surjective, take any $y \in \mathbf{R}$ *and solve for* x *in terms of* y.

$$y = 3 - 4x \implies 4x = 3 - y \implies x = \frac{3 - y}{4}$$

  Since for any $y \in \mathbf{R}$, *there exists an* $x \in \mathbf{R}$, *the function is surjective.*

- ***Conclusion***:

  *The function is both injective and surjective, so it is **bijective**.*

(ii) $f : \mathbf{R} \to \mathbf{R}$ *defined by* $f(x) = 1 + x^2$:

Solution :

- ***Injectivity (One-one)***:

  Assume $f(x_1) = f(x_2)$.

$$1 + x_1^2 = 1 + x_2^2 \implies x_1^2 = x_2^2 \implies x_1 = \pm x_2$$

  Therefore, the function is not injective since x_1 *and* x_2 *can be different but still yield the same function value.*

- ***Surjectivity (Onto)***:

  The range of $f(x) = 1 + x^2$ *is* $[1, \infty)$, *meaning the function only outputs values greater than or equal to 1. Thus, it is not surjective* *onto* $\mathbf{R}$, *since it cannot produce negative values or values less than 1.*

- ***Conclusion***:

  The function is ***neither injective nor surjective***, *so it is not bijective.*

8. *Let* A *and* B *be sets. Show that* $f : A \times B \to B \times A$ *such that* $f(a, b) = (b, a)$ *is a bijective function.*

Solution :

- ***Injectivity (One-one)***:

  Suppose $f(a_1, b_1) = f(a_2, b_2)$. *Then:*

$$(b_1, a_1) = (b_2, a_2)$$

  This implies $b_1 = b_2$ *and* $a_1 = a_2$, *so* $(a_1, b_1) = (a_2, b_2)$. *Hence, the function is injective.*

- ***Surjectivity (Onto)***:

  For any $(b, a) \in B \times A$, *there exists* $(a, b) \in A \times B$ *such that* $f(a, b) = (b, a)$. *Therefore,* f *is surjective.*

- ***Conclusion***:

  Since f *is both injective and surjective, it is* ***bijective***.

9. *Let* $f : \mathbf{N} \to \mathbf{N}$ *be defined by:*

$$f(n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

State whether the function f *is bijective. Justify your answer.*

Solution :

- ***Injectivity (One-one)***:

  If n_1 *and* n_2 *are both odd, or both even, then* $f(n_1) = f(n_2)$ *implies* $n_1 = n_2$. *However, if* n_1 *is odd and* n_2 *is even, their function values are different, so the function is injective.*

- ***Surjectivity (Onto)***:

  For every natural number $m \in \mathbf{N}$, *we can find an* $n \in \mathbf{N}$ *such that* $f(n) = m$. *If* m *is odd,* $n = 2m - 1$; *if* m *is even,* $n = 2m$. *Therefore, the function is surjective.*

- ***Conclusion***:

  Since the function is both injective and surjective, it is ***bijective***.

10. *Let* $A = \mathbf{R} - \{3\}$ *and* $B = \mathbf{R} - \{1\}$. *Consider the function* $f : A \to B$ *defined by* $f(x) = \left(\frac{x-2}{x-3}\right)$. *Is* f *one-one and onto? Justify your answer.*

Solution :

- ***Injectivity (One-one)***:

  Assume $f(x_1) = f(x_2)$.

  $$\frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$$

  Cross-multiply and simplify:

  $$(x_1 - 2)(x_2 - 3) = (x_2 - 2)(x_1 - 3)$$

  Expanding both sides and simplifying, we eventually get $x_1 = x_2$, *so the function is injective.*

- ***Surjectivity (Onto)***:

  For any $y \in B$, *solve* $y = \frac{x-2}{x-3}$ *for* x:

$$y(x-3) = x - 2 \implies yx - 3y = x - 2$$

  Rearranging, $(y-1)x = 3y - 2$, *so:*

$$x = \frac{3y-2}{y-1}$$

  This solution is valid for all $y \neq 1$, *which covers the codomain* $B = \mathbf{R} - \{1\}$. *Hence, the function is surjective.*

- ***Conclusion***:

  The function is both injective and surjective, so it is ***bijective***.

11. *Let* $f : \mathbf{R} \to \mathbf{R}$ *be defined as* $f(x) = x^4$. *Choose the correct answer.*

   (A) f *is one-one onto*

   (B) f *is many-one onto*

   (C) f *is one-one but not onto*

   (D) f *is neither one-one nor onto*

Solution :

- ***Injectivity***:

  The function $f(x) = x^4$ *is not injective because* $f(-x) = f(x)$. *For example,* $f(1) = f(-1) = 1$. *So, it is not one-one.*

- ***Surjectivity***:

  The range of $f(x) = x^4$ *is* $[0, \infty)$, *meaning the function does not* *map to negative values. Thus, it is not surjective.*

- ***Conclusion:***

  *The correct answer is **(D)**: f is **neither one-one nor onto**.*

12. *Let $f : \mathbf{R} \to \mathbf{R}$ be defined as $f(x) = 3x$. Choose the correct answer.*

   (A) f is one-one onto

   (B) f is many-one onto

   (C) f is one-one but not onto

   (D) f is neither one-one nor onto

Solution :

Injectivity (One-one):

To determine if the function is injective, we need to check if $f(x_1) = f(x_2)$ implies $x_1 = x_2$:

$$f(x_1) = f(x_2) \implies 3x_1 = 3x_2 \implies x_1 = x_2$$

Therefore, the function is injective.

Surjectivity (Onto):

To determine if the function is surjective, we need to check if for every $y \in \mathbf{R}$, there exists an $x \in \mathbf{R}$ such that $f(x) = y$:

$$y = 3x \implies x = \frac{y}{3}$$

For every $y \in \mathbf{R}$, $x = \frac{y}{3}$ is a real number, so the function f is surjec-

tive.

***Conclusion**:*

*The function $f(x) = 3x$ is both injective and surjective. Therefore, it is **bijective**, which means it is one-one and onto.*

Thus, the correct answer is:

***(A)** f **is one-one onto**.*